Stability analysis of a noise-induced Hopf bifurcation

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Abstract. We study analytically and numerically the noise-induced transition between an absorbing and
an oscillatory state in a Duffing oscillator subject to multiplicative, Gaussian white noise. We show in a
non-perturbative manner that a stochastic bifurcation occurs when the Lyapunov exponent of the linearised
system becomes positive. We deduce from a simple formula for the Lyapunov exponent the phase diagram
of the stochastic Duffing oscillator. The behaviour of physical observables, such as the oscillator's mean
energy, is studied both close to and far from the bifurcation.

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1 Introduction

The stability of a dynamical system can be strongly affected by the presence of uncontrolled perturbations such as random noise [1]. Noise-induced bifurcations are of primary interest in many fields of applied mathematics and physics and occur in various experimental contexts, as for instance: electronic circuits [2], mechanical oscillators [3], laser physics [1], surface waves [4], thermal and electro-hydrodynamic convection in fluids or liquid crystals [5–7] and diffusion in random media [8]. In nonlinear dynamical systems, the interplay of noise and nonlinearity may produce unusual phenomena: noise can shift the bifurcation threshold from one phase to another [9], it can induce new phase-transitions in spatially-extended systems [10], or create spatial patterns [11] (for a recent review see [12]).

One of the simplest systems that may serve as a paradigm for noise-induced transitions is the nonlinear oscillator with parametric noise [13–15], i.e., with a frequency that fluctuates randomly with time around a given mean value [16,17]. In a recent work [18,19], we studied the time asymptotic behaviour of such an oscillator in the small damping limit. We showed that physical observables grow algebraically with time until the dissipative time scale is reached; then the system settles into an oscillatory state described by a stationary probability distribution function (PDF) of the energy E: the zero equilibrium point ($E = 0$) is unstable in the small damping limit. In the present work, we investigate the stability of the origin for arbitrary damping rate and noise strength.

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For random dynamical systems, it was recognized early on that various 'naive' stability criteria [16,17], obtained by linearizing the dynamical equation around the origin, lead to ambiguous results. (This is in contrast with the deterministic case for which the bifurcation threshold is obtained without ambiguity by studying the eigenstates of the linearized equations [20].) In fact, it was conjectured in [17] and proved in [21] that, in a linear oscillator with arbitrarily small parametric noise, all moments beyond a certain order diverge in the long time limit. Thus, any criterion based on finite mean displacement, momentum, or energy of the linearized dynamical equation is not adequate to insure global stability: the bifurcation threshold of a nonlinear random dynamical system cannot be determined simply from the moments of the linearized system. In practice, the transition point is usually calculated in a perturbative manner by using weak noise expansions [3,9,14,15].

In the present work, we shall obtain in a non-perturbative manner the bifurcation threshold of a stochastic Duffing oscillator with multiplicative, Gaussian white noise. We use a technique described in [18] where the stochastic dynamical equations are expressed in terms of energy-angle variables. Thanks to a detailed analysis of the associated Fokker-Planck equation, we shall show that the bifurcation occurs precisely where the Lyapunov exponent of the *linear* oscillator changes sign. This finding is confirmed by numerical simulations. The stability of the fixed point $E = 0$ of the (nonlinear) Duffing oscillator with parametric noise is deduced, in this sense, from the linearized dynamics. We shall also derive the scaling

behaviour of physical observables in the vicinity of the bifurcation with respect to the distance to threshold. Finally, we study the behaviour of observables far from the bifurcation, in the small damping or strong noise limit.

A rigorous mathematical theory of random dynamical systems has been recently developed and theorems relating the stability of the solution of a stochastic differential equation to the sign of Lyapunov exponents have been proved [22]. In the present work, we do not rely upon the sophisticated tools of the general mathematical theory but follow an intuitive approach, easily accessible to physicists, based upon a factorisation hypothesis of the stationary PDF in the limit of small and large energies.

This article is organized as follows. In Section 2, we define the system considered, present the relevant phenomenology, and introduce energy-angle variables for the stochastic Duffing oscillator. We study the stationary PDF of the energy in Section 3. In Section 4, we derive the stability criterion for the stochastic Duffing oscillator and obtain its phase diagram. In Section 5, we perform a local analysis of the physical observables in the vicinity of and far from the bifurcation threshold. Section 6 is devoted to some concluding remarks. In Appendix A, we prove some useful identities satisfied by the Lyapunov exponent. In Appendix B, an explicit formula for the Lyapunov exponent is derived.

2 The Duffing oscillator with multiplicative white noise

2.1 Notations and phenomenology

A nonlinear oscillator with a randomly varying frequency due to external noise can be described by the following equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x}{\mathrm{d}t} + (\omega^2 + \xi(t))x + \frac{\partial \mathcal{U}(x)}{\partial x} = 0, \qquad (1)$$

where $x(t)$ is the position of the oscillator at time t, $\mathcal{U}(x)$ an anharmonic confining potential and γ the (positive) friction coefficient. The linear frequency of the oscillator has a mean value ω and its fluctuations are modeled by a Gaussian white noise $\xi(t)$ of zero mean-value and of amplitude $\mathcal{D}$

$$\begin{aligned} \langle \xi(t) \rangle &= 0 \,, \\ \langle \xi(t)\xi(t') \rangle &= \mathcal{D}\,\delta(t - t') \,. \end{aligned} \qquad (2)$$

In this work, all stochastic differential equations are interpreted according to Stratonovich calculus. We shall study the Duffing oscillator with multiplicative noise, the confining potential $\mathcal{U}$ being given by

$$\mathcal{U}(x) = \lambda \frac{x^4}{4} \,. \qquad (3)$$

By rewriting time and amplitude in dimensionless units, $t := \omega t$ and $x := \lambda^{1/2}\omega^{-1}x$, respectively, equation (1) becomes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \alpha \frac{\mathrm{d}x}{\mathrm{d}t} + x + x^3 = x\,\Xi(t), \qquad (4)$$

where $\Xi(t)$ is a delta-correlated Gaussian variable:

$$\begin{aligned} \langle \Xi(t) \rangle &= 0 \,, \\ \langle \Xi(t)\Xi(t') \rangle &= \Delta\,\delta(t - t') \,. \end{aligned} \qquad (5)$$

The parameters

$$\alpha = \gamma/\omega \quad \text{and} \quad \Delta = \mathcal{D}/\omega^3 \qquad (6)$$

correspond to dimensionless dissipation rate and to noise strength, respectively. In the rest of this work, all physical quantities are expressed in dimensionless units.

In Figure 1, we present, for three values of the parameters, a trajectory in the phase plane $(x, v = \dot{x})$ characteristic of the oscillator's behaviour. Here as well as for other simulations presented in this article, equation (4) is integrated numerically by using the one step collocation method advocated in [23] and described in detail in [18]. Initial conditions are chosen far from the origin, with an amplitude of order 1. Noisy oscillations are observed for small values of the damping parameter α (see Fig. 1a). A larger value of α turns the origin into a global attractor for the dynamics (see Fig. 1b). Increasing the noise amplitude Δ at constant α makes the origin unstable again (see Fig. 1c).

For all values of the parameters (α, Δ), the origin $(x = \dot{x} = 0)$ is a fixed point of the random dynamical system (4). Our main goal, in this work, is to investigate the stability properties of the fixed point at the origin when the control parameters vary.

2.2 Energy-angle representation of the Duffing oscillator

We now follow the method presented in [18] and introduce the energy-angle variables associated with the Duffing oscillator. The deterministic and undamped oscillator corresponding to equation (4) is an integrable dynamical system because the mechanical energy E, defined as

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 \,, \qquad (7)$$

is a conserved quantity. Introducing the angular variable ϕ, the constant energy orbits of the Hamiltonian system $\ddot{x} + x + x^3 = 0$ are parametrized as

$$x = \left(\frac{4E^2}{4E + 1} \right)^{1/4} \mathrm{sd}\,(\phi\,;k) \,, \qquad (8)$$

$$\dot{x} = (2E)^{1/2} \frac{\mathrm{cn}\,(\phi\,;k)}{\mathrm{dn}^2\,(\phi\,;k)}, \qquad (9)$$

where the functions sd, cn and dn are Jacobi elliptic functions [24]. The elliptic modulus k is a function of the energy and is given by

$$k^2 = \frac{\sqrt{4E + 1} - 1}{2\sqrt{4E + 1}} \,. \qquad (10)$$
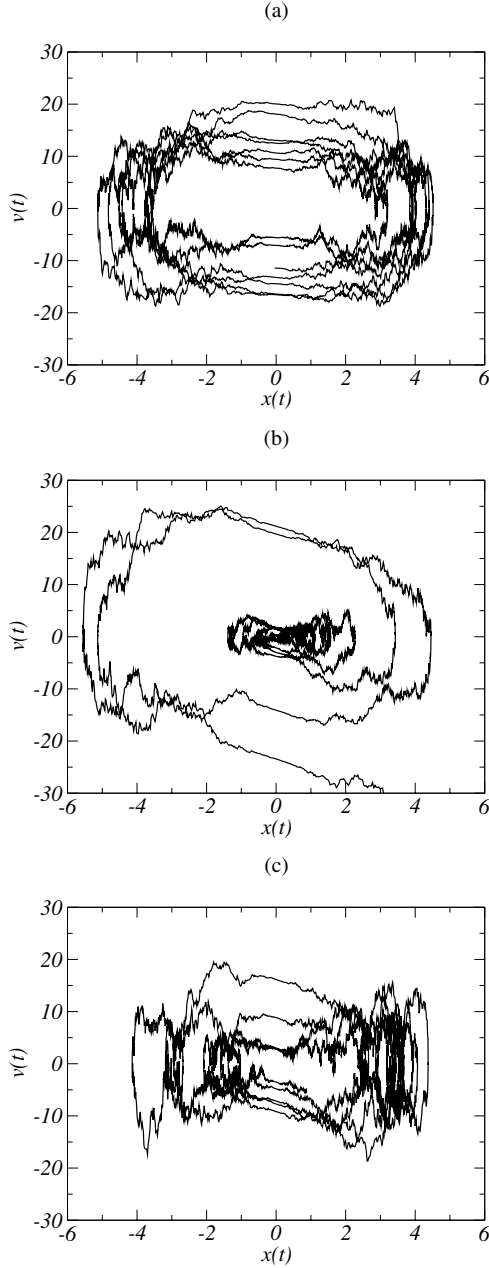
(a)

(b)

(c)

Fig. 1. Phase plane plots of a typical trajectory of the noisy Duffing oscillator. Equation (4) is integrated numerically with a time step $\delta t = 5 \times 10^{-4}$. (a) $\alpha = 0.5$, $\Delta = 20$, $10 \leq t \leq 100$; (b) $\alpha = 2.0$, $\Delta = 20$, $t \leq 100$; (c) $\alpha = 2.0$, $\Delta = 60$, $10 \leq t \leq 100$. Early-time relaxation towards noisy oscillations has been omitted for clarity in cases (a) and (c). The trajectory spirals down towards the globally stable origin in case (b).

In the (E, ϕ) coordinate system, the dynamical equations for the Duffing oscillator, without noise and dissipation, become simply

$$\dot{E} = 0 \, ,$$
$$\dot{\phi} = (4E + 1)^{\frac{1}{4}} \, . \tag{11}$$

We reintroduce the multiplicative noise and the dissipation terms, and obtain a system of coupled stochastic

equations that describes the evolution of the energy and angle variables:

$$\dot{E} = J_E(E, \phi) + D_E(E, \phi)\, \Xi(t) \, , \tag{12}$$
$$\dot{\phi} = J_\phi(E, \phi) + D_\phi(E, \phi)\, \Xi(t) \, , \tag{13}$$

where the drift and diffusion coefficients J_E, J_ϕ, D_E and D_ϕ are defined by (see [18] for details of the derivation):

$$J_E(E, \phi) = -2\,\alpha\, E\, \frac{\text{cn}^2(\phi; k)}{\text{dn}^4(\phi; k)} \tag{14}$$

$$D_E(E, \phi) = \frac{2E}{(4E + 1)^{\frac{1}{4}}}\, \frac{\text{sn}(\phi; k)\text{cn}(\phi; k)}{\text{dn}^3(\phi; k)} \tag{15}$$

$$J_\phi(E, \phi) = (4E + 1)^{\frac{1}{4}} + \alpha\, \frac{2E + 1}{4E + 1}\, \frac{\text{sn}(\phi; k)\text{cn}(\phi; k)}{\text{dn}^3(\phi; k)} \tag{16}$$

$$\begin{aligned}
D_\phi(E, \phi) = &-\frac{(2E + 1)}{(4E + 1)^{\frac{5}{4}}}\, \text{sd}^2(\phi; k) \\
&- \frac{E}{(4E + 1)^{\frac{7}{4}}}\text{sd}^4(\phi; k) + \frac{\alpha\, E}{(4E + 1)^{\frac{3}{2}}} \\
&\times \frac{\text{cn}(\phi; k)\text{sd}^3(\phi; k)}{\text{dn}^2(\phi; k)} + \frac{E}{(4E + 1)^{\frac{7}{4}}} \\
&\times \frac{\text{sn}(\phi; k)\text{cn}(\phi; k)}{\text{dn}^3(\phi; k)} \int_0^\phi \text{sd}^2(\theta; k)d\theta \\
&- \frac{\alpha\, E}{(4E + 1)^{\frac{3}{2}}}\, \frac{\text{cn}^2(\phi; k)}{\text{dn}^4(\phi; k)} \int_0^\phi \text{sd}^2(\theta; k)d\theta.
\end{aligned} \tag{17}$$

3 Analysis of the Fokker-Planck equation

Using the same notations, the Fokker-Planck equation that governs the time evolution of the probability distribution function $P_t(E, \phi)$ associated with the stochastic differential system (12–13) reads [25]

$$\begin{aligned}
\partial_t P_t(E, \phi) = &-\partial_E\,(J_E P) - \partial_\phi\,(J_\phi P) \\
&+ \frac{\Delta}{2}\big[\partial_E\,(D_E \partial_E(D_E P)) + \partial_E\,(D_E \partial_\phi(D_\phi P)) \\
&+ \partial_\phi\,(D_\phi \partial_E(D_E P)) + \partial_\phi\,(D_\phi \partial_\phi(D_\phi P))\,\big]. \tag{18}
\end{aligned}$$

The goal of this section is to determine the conditions under which there exist stationary solutions $P_{\text{stat}}(E, \phi)$ of equation (18) with a non-zero mean energy. Such an extended PDF describes an oscillatory asymptotic state.

Obtaining an analytical expression of $P_{\text{stat}}(E, \phi)$ in the general case is a daunting task. In the following, we shall perform a local analysis of equation (18) for small and large values of E. In these limiting cases, the expressions of the drift and diffusion coefficients take simpler forms. We may write, in full generality,

$$P_{\text{stat}}(E, \phi) = P_{\text{stat}}(\phi|E)\, \tilde{P}_{\text{stat}}(E) \, , \tag{19}$$

where $P_{\text{stat}}(\phi|E)$ represents, in the stationary state, the conditional probability of the angle ϕ at a fixed value

of the energy E. Assuming the conditional probability $P_{\text{stat}}(\phi|E)$ to be independent of E when $E \ll 1$ and $E \gg 1$ (respectively Sects. 3.1 and 3.2), we shall derive the local behaviour of a stationary solution of the Fokker-Planck equation (18). This solution is a legitimate PDF when it is normalizable: the normalizability condition provides the location of the bifurcation [26].

3.1 Small E limit: the linearized stochastic oscillator

When the oscillator's mechanical energy E is small, the position $x(t)$ and the velocity $\dot{x}(t)$ are also small: nonlinear terms may be neglected. The Duffing oscillator simplifies to a linear (harmonic) oscillator:

$$\ddot{x} + \alpha\dot{x} + x = x\,\Xi(t) \tag{20}$$

with $\langle \Xi(t)\Xi(t')\rangle = \Delta\,\delta(t-t')$. The energy now reads $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$, and equations (8) and (9) reduce to

$$x = \sqrt{2E}\sin\phi\,, \tag{21}$$

$$\dot{x} = \sqrt{2E}\cos\phi\,. \tag{22}$$

In the small E limit, the elliptic modulus k goes to zero, the elliptic functions reduce to circular functions [24], and the drift and diffusion coefficients simplify to

$$J_E(E,\phi) = -2\,\alpha\,E\cos^2\phi \tag{23}$$

$$D_E(E,\phi) = 2\,E\sin\phi\cos\phi\,, \tag{24}$$

$$J_\phi(E,\phi) = 1 + \alpha\sin\phi\cos\phi\,, \tag{25}$$

$$D_\phi(E,\phi) = -\sin^2\phi\,. \tag{26}$$

Substituting the expressions (23–26) in equation (18) and integrating over E, we obtain an independent Fokker-Planck equation for the marginal PDF $\tilde{P}_t(\phi)$:

$$\partial_t\tilde{P}_t(\phi) = -\partial_\phi\left[(1 + \alpha\sin\phi\cos\phi)\,\tilde{P}_t(\phi)\right]$$
$$+ \frac{\Delta}{2}\partial_\phi\left[\sin^2\phi\,\partial_\phi(\sin^2\phi\,\tilde{P}_t(\phi))\right]. \tag{27}$$

An explicit formula for the stationary angular measure $\tilde{P}_{\text{stat}}(\phi)$, solution of equation (27), is derived in Appendix B.

We now make the hypothesis that in the small E limit, the stationary conditional measure $P_{\text{stat}}(\phi|E)$ becomes independent of E and is identical to the stationary angular distribution $\tilde{P}_{\text{stat}}(\phi)$ of the harmonic oscillator:

$$\text{for}\quad E \ll 1\,,\quad P_{\text{stat}}(\phi|E) = \tilde{P}_{\text{stat}}(\phi)\,. \tag{28}$$

Equation (19) becomes

$$P_{\text{stat}}(E,\phi) = \tilde{P}_{\text{stat}}(E)\tilde{P}_{\text{stat}}(\phi)\,, \tag{29}$$

i.e., we assume that the stationary PDF factorizes when $E \ll 1$. An exactly solvable equation for $\tilde{P}_{\text{stat}}(E)$ can now be obtained by inserting expressions (23–26) and the

identity (29) in the Fokker-Planck equation (18) and then averaging over ϕ:

$$\frac{\Delta}{2}\left\langle\sin^2\phi\cos^2\phi\right\rangle\partial_E\left(E\tilde{P}_{\text{stat}}(E)\right) =$$
$$\left(\frac{\Delta}{2}\left\langle\sin^2\phi\left(\sin^2\phi - \cos^2\phi\right)\right\rangle - \alpha\left\langle\cos^2\phi\right\rangle\right)\tilde{P}_{\text{stat}}(E)\,. \tag{30}$$

Expectation values denoted by brackets $\langle\;\rangle$ are calculated using the stationary angular measure $\tilde{P}_{\text{stat}}(\phi)$. Introducing the notations:

$$\mu(\alpha,\Delta) = \frac{\Delta}{2}\left\langle\sin^2\phi\cos^2\phi\right\rangle\,, \tag{31}$$

$$\Lambda(\alpha,\Delta) = \frac{\Delta}{2}\left\langle\sin^2\phi\left(\sin^2\phi - \cos^2\phi\right)\right\rangle - \alpha\left\langle\cos^2\phi\right\rangle\,, \tag{32}$$

equation (30) becomes:

$$\mu\,\partial_E\left(E\tilde{P}_{\text{stat}}(E)\right) = \Lambda\,\tilde{P}_{\text{stat}}(E)\,, \tag{33}$$

and admits the solution:

$$\tilde{P}_{\text{stat}}(E) \propto E^{\frac{\Lambda}{\mu}-1}\quad\text{for}\quad E \ll 1\,. \tag{34}$$

Thus, for $E \ll 1$, the PDF $\tilde{P}_{\text{stat}}(E)$ behaves as a power law of the energy.

The coefficient μ is always positive, but the sign of Λ is a function of the parameters α and Δ. For negative values of Λ, the stationary solution (34) is not normalizable and cannot represent a PDF; the only stationary PDF solution of (18) is then a delta function centered at the origin:

$$\tilde{P}_{\text{stat}}(E) = \delta(E)\quad\text{for}\quad\Lambda < 0\,. \tag{35}$$

Thus for negative values of Λ, the origin is a global attractor for the stochastic nonlinear oscillator. Non-trivial stationary solutions exist only for $\Lambda > 0$: the bifurcation threshold corresponds to values of the parameters for which $\Lambda(\alpha,\Delta) = 0$.

3.2 Large E limit

For very large values of E, the elliptic modulus k, defined in equation (10), tends to $1/\sqrt{2}$, and the drift and diffusion coefficients become

$$J_E(E,\phi) = -2\,\alpha\,E\,\frac{\text{cn}^2(\phi)}{\text{dn}^4(\phi)}, \tag{36}$$

$$D_E(E,\phi) = \sqrt{2}\,E^{3/4}\,\frac{\text{sn}(\phi)\,\text{cn}(\phi)}{\text{dn}^3(\phi)}, \tag{37}$$

$$J_\phi(E,\phi) = (4E)^{1/4} + \frac{\alpha}{2}\,\frac{\text{sn}(\phi)\,\text{cn}(\phi)}{\text{dn}^3(\phi)}, \tag{38}$$

$$D_\phi(E,\phi) = -\frac{1}{2\sqrt{2}E^{1/4}}\,\text{sd}^2(\phi), \tag{39}$$

in agreement with equations (3, 4) of [19] (we have omitted the elliptic modulus $1/\sqrt{2}$ for sake of simplicity). Using expressions (36) to (39) for the drift and diffusion coefficients, the stochastic equations (12, 13) reduce, in the large E limit, to

$$\dot{E} = -2\,\alpha\,E\,\frac{\mathrm{cn}^2(\phi)}{\mathrm{dn}^4(\phi)} + \sqrt{2}\,E^{3/4}\,\frac{\mathrm{sn}(\phi)\,\mathrm{cn}(\phi)}{\mathrm{dn}^3(\phi)}\,\Xi(t) \qquad (40)$$

$$\dot{\phi} = (4E)^{1/4} + \frac{\alpha}{2}\frac{\mathrm{sn}(\phi)\,\mathrm{cn}(\phi)}{\mathrm{dn}^3(\phi)} - \frac{\sqrt{2}}{4E^{1/4}}\mathrm{sd}^2(\phi)\,\Xi(\mathrm{t}). \quad (41)$$

In terms of the variable $\Omega = E^{1/4}$, equation (40) becomes linear and has a form similar to Ornstein-Uhlenbeck's equation. Because the elliptic functions are bounded, Ω saturates to a value of the order of $\frac{\Delta}{\alpha}$. Thus, we deduce from equation (41) that the phase ϕ grows linearly with time. Hence, ϕ is a fast variable as compared to Ω. Integrating out this fast variable from the Fokker-Planck equation leads to an effective stochastic dynamics for Ω, as explained in detail in [18]. An equivalent formulation is to assume that, in the large E limit, the stationary conditional probability $P_{\mathrm{stat}}(\phi|E)$, defined in equation (19), becomes uniform in ϕ and independent of E, i.e.,

$$\text{for} \quad E \gg 1, \quad P_{\mathrm{stat}}(\phi\,|\,E) \sim 1. \qquad (42)$$

Averaging the dynamics over ϕ yields the following expression for the stationary PDF [13,19]:

$$\text{for } E \gg 1, \tilde{P}_{\mathrm{stat}}(E) \propto E^{-1/4} \exp\left\{-\frac{8\,\alpha\,E^{1/2}}{3\,\mathcal{M}\,\Delta}\right\}, \quad (43)$$

with $\mathcal{M} = \frac{8}{5\pi^2}\Gamma\left(3/4\right)^4 \simeq 0.3655$, Γ being the Eulerian factorial function. A stationary solution of (18) decays as a stretched exponential and is always integrable at infinity irrespective of the values of α and Δ.

4 Phase diagram of the Duffing oscillator

In the long time limit, the energy of a harmonic oscillator with parametric noise (20) varies exponentially with time [16]. This behaviour is characterized by a Lyapunov exponent, defined as $\lim_{t\to\infty}\frac{1}{2t}\overline{\log E}$, where the overline means averaging over $P_t(E,\phi)$ [27]. (The factor $1/2$ is included for consistency with the usual definition of the Lyapunov exponent in terms of the position $x(t)$.)

In Appendix A, we prove that the coefficient $\Lambda(\alpha, \Delta)$ defined in equation (32) is equal to the Lyapunov exponent:

$$\Lambda(\alpha, \Delta) = \lim_{t\to\infty}\ \frac{1}{2t}\ \overline{\log E}. \qquad (44)$$

For the (nonlinear) Duffing oscillator, we found in Section 3 that a non-trivial stationary solution $P_{\mathrm{stat}}(E,\phi)$ of equation (18) exists only when the Lyapunov exponent $\Lambda(\alpha, \Delta)$ of the associated linear oscillator is positive. This extended PDF describes an oscillatory asymptotic state
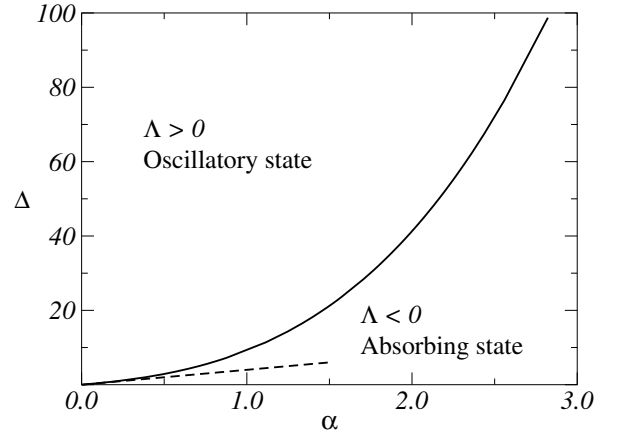

Fig. 2. Phase diagram of the Duffing oscillator with multiplicative noise. The critical curve separates an absorbing and an oscillatory phase in the (α, Δ) plane. The dashed line represents the asymptotic approximation, equation (47).

for the Duffing oscillator, as opposed to the absorbing state associated with the delta function (35). The location of the stochastic bifurcation threshold of the Duffing oscillator, which separates the two states, is given by the curve $\Lambda(\alpha, \Delta) = 0$. The following explicit formula is derived in Appendix B:

$$\Lambda(\alpha,\Delta) = \frac{1}{2}\left\{\frac{\int_0^{+\infty}\mathrm{d}u\ \sqrt{u}\ e^{-\frac{2}{\Delta}\left((1-\frac{\alpha^2}{4})u+\frac{u^3}{12}\right)}}{\int_0^{+\infty}\frac{\mathrm{d}u}{\sqrt{u}}e^{-\frac{2}{\Delta}\left((1-\frac{\alpha^2}{4})u+\frac{u^3}{12}\right)}} - \alpha\right\}. \qquad (45)$$

The transition line is determined by the relation between α and Δ for which the Lyapunov exponent vanishes. For a given value of Δ, the critical value of the damping $\alpha = \alpha_c(\Delta)$ satisfies

$$\alpha_c = \frac{\int_0^{+\infty}\mathrm{d}u\ \sqrt{u}\ \exp\left\{-\frac{2}{\Delta}\left((1-\frac{\alpha_c^2}{4})u+\frac{u^3}{12}\right)\right\}}{\int_0^{+\infty}\frac{\mathrm{d}u}{\sqrt{u}}\exp\left\{-\frac{2}{\Delta}\left((1-\frac{\alpha_c^2}{4})u+\frac{u^3}{12}\right)\right\}}. \quad (46)$$

The critical curve $\alpha = \alpha_c(\Delta)$ (Eq. (46)) is represented in Figure 2. It separates two regions in parameter space: for $\alpha < \alpha_c$ (resp. $\alpha > \alpha_c$) the Lyapunov exponent is positive (resp. negative), the stationary PDF is an extended function (resp. a delta distribution) of the energy, the origin is unstable (resp. stable). For a detailed analytical study, we must now distinguish three cases corresponding to an underdamped ($\alpha_c < 2$), critically damped ($\alpha_c = 2$) and overdamped ($\alpha_c > 2$) oscillator.

4.1 Underdamped oscillator

The case $\alpha_c < 2$ can formally be mapped onto that of a linear oscillator with no damping. In terms of the variables

$v = u \left[1 - \frac{\alpha_c^2}{4}\right]^{-1/2}$ and $\Delta_< = \Delta \left[1 - \frac{\alpha_c^2}{4}\right]^{-3/2}$, equation (46) reads

$$\alpha_c = \left[1 - \frac{\alpha_c^2}{4}\right]^{1/2} \frac{\int_0^{+\infty} dv \sqrt{v} \exp\left\{-\frac{2}{\Delta_<}\left(v + \frac{v^3}{12}\right)\right\}}{\int_0^{+\infty} \frac{dv}{\sqrt{v}} \exp\left\{-\frac{2}{\Delta_<}\left(v + \frac{v^3}{12}\right)\right\}}$$

$$= \frac{\Delta_<}{2}\left[1 - \frac{\alpha_c^2}{4}\right]^{1/2} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, \Gamma\left(3n + \frac{3}{2}\right)\left(\frac{\Delta_<^2}{48}\right)^n}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, \Gamma\left(3n + \frac{1}{2}\right)\left(\frac{\Delta_<^2}{48}\right)^n}.$$

Retaining only the lowest order term in this series, we obtain the limiting behaviour of the critical curve $\alpha_c(\Delta)$, for small values of α and Δ:

$$\alpha_c = \frac{\Delta}{4}. \qquad (47)$$

The change of variables used here breaks down at $\alpha_c = 2$, we shall therefore consider this case separately.

4.2 Critically damped oscillator

For a critically damped oscillator, we substitute $\alpha_c = 2$ in equation (46) and obtain

$$2 = \frac{\int_0^{+\infty} du \sqrt{u} \exp\left(-\frac{u^3}{6\Delta_c}\right)}{\int_0^{+\infty} \frac{du}{\sqrt{u}} \exp\left(-\frac{u^3}{6\Delta_c}\right)} = (6\Delta_c)^{1/3} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{6}\right)}. \qquad (48)$$

Thus, the critical noise amplitude Δ_c for $\alpha_c = 2$ is

$$\Delta_c = \frac{4}{3} \frac{\Gamma\left(\frac{1}{6}\right)^3}{\pi^{3/2}} \simeq 41.2969..., \qquad (49)$$

and the expression (45) for the Lyapunov exponent reduces to

$$\Lambda = \left(\frac{\Delta}{\Delta_c}\right)^{1/3} - 1. \qquad (50)$$

4.3 Overdamped oscillator

For $\alpha_c > 2$, we define $v = u \left[\frac{\alpha_c^2}{4} - 1\right]^{-1/2}$ and $\Delta_> = \Delta \left[\frac{\alpha_c^2}{4} - 1\right]^{-3/2}$. Equation (46) then becomes

$$\alpha_c \left[\frac{\alpha_c^2}{4} - 1\right]^{-1/2} = \frac{\int_0^{+\infty} dv \sqrt{v}\, e^{-\frac{2}{\Delta_>}\left(-v + \frac{v^3}{12}\right)}}{\int_0^{+\infty} \frac{dv}{\sqrt{v}} e^{-\frac{2}{\Delta_>}\left(-v + \frac{v^3}{12}\right)}}$$

$$= (6\Delta_>)^{1/3} \frac{\sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{2n+3}{6}\right)\left(\frac{48}{\Delta_>^2}\right)^{n/3}}{\sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{2n+1}{6}\right)\left(\frac{48}{\Delta_>^2}\right)^{n/3}}. \qquad (51)$$

When $\alpha_c \to \infty$, the left hand side of equation (51) converges to 2 and $\Delta_>$ tends to a limiting value $\mathcal{K}$ that we computed numerically: $\mathcal{K} \simeq 28.337$. Thus, in the large α_c limit, we find the following relation between α_c and Δ

$$\Delta \simeq \mathcal{K} \left[\frac{\alpha_c^2}{4} - 1\right]^{3/2} \simeq \frac{\mathcal{K}}{8} \alpha_c^3 \simeq 3.54\, \alpha_c^3. \qquad (52)$$

When $\alpha_c \to 2^+$, the left hand side of equation (51) increases to infinity. Retaining only the leading order term of the series, we obtain

$$2 \left[\frac{\alpha_c^2}{4} - 1\right]^{-1/2} = (6\Delta_>)^{1/3} \frac{\Gamma(1/2)}{\Gamma(1/6)}. \qquad (53)$$

This equation is consistent with equation (49).

5 Physical observables

The goal of this section is to determine how physical observables, such as the energy and the position of the oscillator, scale near the bifurcation threshold as well as far from it. The agreement between analytical calculations and numerical simulations will validate the assumptions made in Section 3 concerning the behaviour of the conditional PDF $P_{\text{stat}}(\phi|E)$ for small and large values of E.

Let us first of all review the results obtained in Section 3. For parameter values such that $\Lambda < 0$, we demonstrated that $\tilde{P}_{\text{stat}}(E) = \delta(E)$: the origin is a global attractor of the dynamics. Conversely, when $\Lambda > 0$, the origin becomes unstable. The stationary measure of the energy is then an extended function: it behaves as a power law for small values of E (see Eq. (34)), and decreases as a stretched exponential for large values of E (see Eq. (43)):

$$\text{for } E \ll 1, \ \tilde{P}_{\text{stat}}^<(E) = A E^{\frac{\Lambda}{\mu} - 1},$$

$$\text{for } E \gg 1, \ \tilde{P}_{\text{stat}}^>(E) = B E^{-\frac{1}{4}} \exp\left\{-\frac{8 \alpha E^{1/2}}{3 \mathcal{M} \Delta}\right\}. \qquad (54)$$

The precise values of A and B can only be obtained by solving the full Fokker-Planck equation (18), and matching the two asymptotic expressions in the intermediate regime $E \simeq 1$.

5.1 Behaviour in the vicinity of the bifurcation

For a given value of the damping rate α, let us choose the noise strength Δ such that $0 < \Delta - \Delta_c(\alpha) \ll 1$. In this case, the parameters of the system are tuned just above the bifurcation threshold and the Lyapunov exponent is slightly positive: $0 < \Lambda \ll 1$. The stationary PDF is an extended function but, as can be inferred from equation (54), most of its mass is concentrated in the vicinity of $E = 0$. Therefore, a good approximation of the stationary PDF is obtained by taking it equal to $\tilde{P}_{\text{stat}}^<(E)$ over a finite interval in the vicinity of 0 (e.g., $0 \leq E \leq E_0$), and equal to
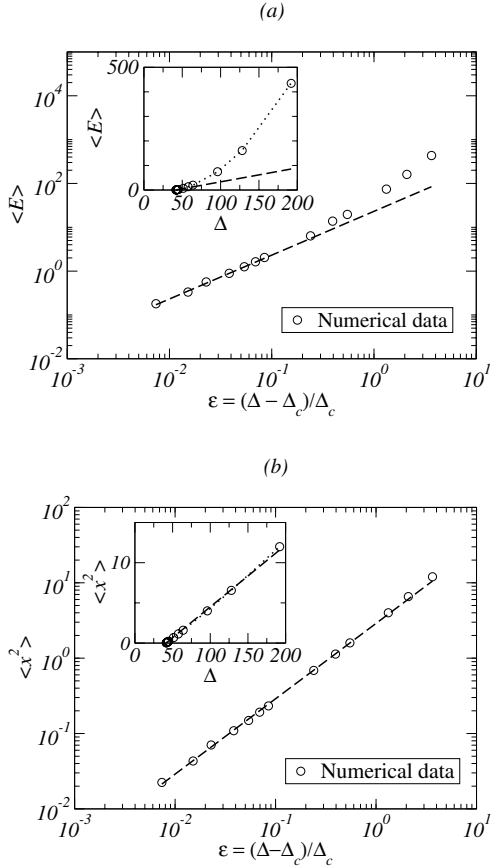
(a)

(b)

Fig. 3. Scaling behaviour of physical observables of the stochastic Duffing oscillator in the vicinity of the bifurcation threshold as a function of the reduced control parameter $\varepsilon = (\Delta - \Delta_c)/\Delta_c$. (a) Average energy; (b) Variance of the position. Dashed lines corresponding to a linear function of ε are drawn to guide the eye. The corresponding linear-linear graphs are shown as insets. Equation (4) is integrated numerically for $\alpha = 2$, with a time step $\delta t = 5 \times 10^{-4}$. The threshold value is given by equation (49): $\Delta_c(2) \simeq 41.3$. Ensemble averages are computed over 10^3 realisations.

zero outside this interval. The mean value of the energy is given by:

$$\langle E \rangle_{\text{stat}} \simeq \frac{\int_0^{E_0} E^{\frac{\Delta}{\mu}} \mathrm{d}E}{\int_0^{E_0} E^{\frac{\Delta}{\mu}-1} \mathrm{d}E} \propto E_0 \, \frac{\Lambda}{\mu}\,. \qquad (55)$$

If we now take into account that: (i) $\Lambda(\alpha, \Delta_c(\alpha)) = 0$; and (ii) the Lyapunov exponent and the coefficient μ are regular functions of the noise strength Δ when $\Delta \neq 0$, we obtain from equation (55)

$$\langle E \rangle_{\text{stat}} \propto \Lambda(\alpha, \Delta) - \Lambda(\alpha, \Delta_c) \propto (\Delta - \Delta_c)\,, \qquad (56)$$

where the last identity is obtained by a Taylor expansion of Λ for Δ close to Δ_c. This result is confirmed by numerical simulations (see Fig. 3a). A similar reasoning shows that $\langle E \rangle_{\text{stat}}$ scales linearly with $\alpha - \alpha_c(\Delta)$ at fixed noise strength Δ.

Let us now consider the variance of the oscillator's position close to the bifurcation. Using equations (21)
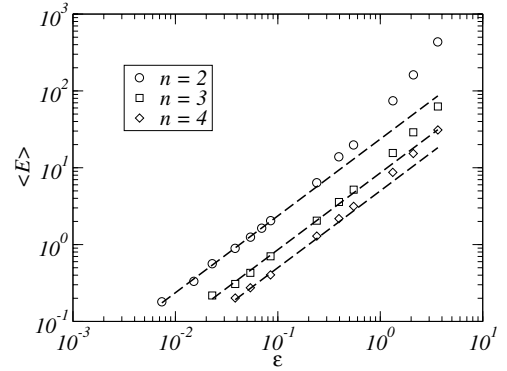

Fig. 4. Scaling behaviour of the mean energy of the general nonlinear oscillator in the vicinity of the bifurcation threshold as a function of the reduced control parameter $\varepsilon = (\Delta - \Delta_c)/\Delta_c$. Dashed lines indicating a linear dependence in ε are drawn to guide the eye. Equation (1) is integrated numerically for $\alpha = 2$, and with $\mathcal{U}(x) = x^{2n}/(2n)$, $n = 2$, 3 and 4. Data obtained for $n = 2$ are the same as in Figure 3. The time step is $\delta t = 5 \times 10^{-4}$; ensemble averages are computed over 10^3 realisations.

and (29), we find

$$\begin{aligned}
\langle x^2 \rangle_{\text{stat}} &= \iint P_{\text{stat}}(E, \phi)\, 2E \sin^2 \phi \, \mathrm{d}E \, \mathrm{d}\phi \\
&= 2 \langle \sin^2 \phi \rangle \langle E \rangle_{\text{stat}} \\
&\propto (\Delta - \Delta_c)\,.
\end{aligned} \qquad (57)$$

The variance of the position scales linearly with the distance to threshold (see Fig. 3b). Using a similar argument, one also finds that $\langle v^2 \rangle_{\text{stat}} \propto (\Delta - \Delta_c)$.

As seen in Figure 4 the linear scaling (56) of the energy with respect to the distance to threshold is observed for $\mathcal{U}(x) \propto x^{2n}$, $n = 2$, 3 and 4, where $\Delta_c(\alpha)$ is *always* given by $\Lambda(\alpha, \Delta_c(\alpha)) = 0$. The threshold is independent of the nonlinearity involved in the confining potential $\mathcal{U}$ appearing in equation (1), because the local expression $\tilde{P}_{\text{stat}}^<(E)$ of the stationary PDF in the vicinity of $E = 0$ is obtained from the *linearized* stochastic differential equation. The observed dependence of the constant of proportionality in equation (56) upon the stiffness of the anharmonic potential remains to be understood.

5.2 Scaling far from the bifurcation

We shall now study the scaling of the physical observables far from the bifurcation point. This situation is encountered, for instance, when the damping rate is vanishingly small for a given noise strength. The Lyapunov exponent is then a finite, strictly positive, number. When $\alpha \to 0$, most of the mass of the stationary PDF is concentrated at large values of E as can be seen from equation (54). Thus, a satisfactory approximation of the stationary PDF is to assume it to be identical to the expression $\tilde{P}_{\text{stat}}^>(E)$
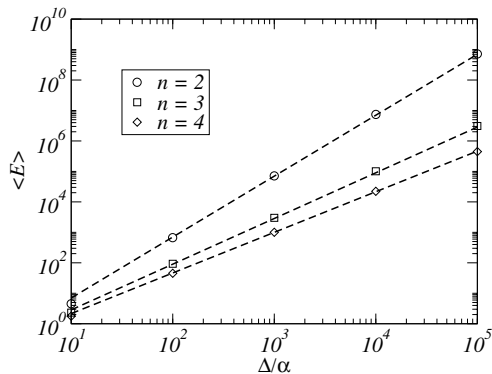
Fig. 5. Scaling behaviour of the mean energy of the general nonlinear oscillator far from the bifurcation as a function of the parameter Δ/α. Equation (1) is integrated numerically with $\mathcal{U}(x) = x^{2n}/(2n)$, $n = 2$, 3 and 4. Dashed lines correspond to the predicted scaling (60), with a scaling exponent $n/(n-1) = 2$, 3/2 and 4/3. The time step is $\delta t = 5 \times 10^{-4}$; ensemble averages are computed over 10^3 realisations.

for all E. After a suitable normalisation, we find

$$\tilde{P}_{\text{stat}}(E) = \frac{1}{2E\Gamma(3/2)} \left(\frac{8\,\alpha\,E^{1/2}}{3\,\mathcal{M}\,\Delta} \right)^{3/2} \exp\left\{ -\frac{8\,\alpha\,E^{1/2}}{3\,\mathcal{M}\,\Delta} \right\},$$

$$(58)$$

with the numerical factor $\mathcal{M}$ defined in equation (43). In this low damping regime, we recover, as expected, the results derived in references [18,19]. The average energy is readily obtained from equation (58)

$$\langle E \rangle_{\text{stat}} = \frac{135}{256}\, \mathcal{M}^2 \left(\frac{\Delta}{\alpha} \right)^2 \simeq 0.070 \left(\frac{\Delta}{\alpha} \right)^2. \qquad (59)$$

This result is at variance with the usual equilibrium fluctuation-dissipation theorem ($\langle E \rangle_{\text{eq}} \propto \Delta/\alpha$); yet this is not a contradiction since equation (4) models a system out of equilibrium.

Far from the bifurcation threshold, the scaling behaviour depends strongly on the stiffness of the potential $\mathcal{U}$. If we take $\mathcal{U}(x) \propto x^{2n}$, equation (59) can be generalized to [19]

$$\langle E \rangle_{\text{stat}} = \frac{1}{2n}\, \frac{\Gamma\left(\frac{3n+1}{2(n-1)} \right)}{\Gamma\left(\frac{n+1}{2(n-1)} \right)}$$

$$\times \left((n-1)\, \frac{\Gamma\left(\frac{3}{2n} \right)\, \Gamma\left(\frac{3n+1}{2n} \right)}{\Gamma\left(\frac{1}{2n} \right)\, \Gamma\left(\frac{3n+3}{2n} \right)} \right)^{\frac{n}{n-1}} \left(\frac{\Delta}{\alpha} \right)^{\frac{n}{n-1}}. \quad (60)$$

This prediction is again confirmed by numerical simulations (see Fig. 5).

6 Conclusion

For a deterministic Duffing oscillator with linear damping, the fixed point at the origin ($x = \dot{x} = 0$) is a global attractor: irrespective of the choice of initial conditions, the

mechanical energy is totally dissipated in the long time limit and the system converges to the origin. This picture is no longer valid once the oscillator is subject to parametric noise. Indeed, the dissipative Duffing oscillator with parametric white noise undergoes a noise-induced bifurcation between: (i) an absorbing state reminiscent of the deterministic system; and (ii) an oscillatory state, resulting from a dynamical balance between energy dissipation and (noise-induced) energy injection. Noisy oscillations appear when the fixed point at the origin becomes unstable: this transition is thus referred to as a (forward) stochastic Hopf bifurcation. Both states may be characterized by the stationary probability distribution function of the oscillator's mechanical energy E. The PDF of the absorbing state is a delta-function, whereas the oscillatory state corresponds to an extended PDF.

Our main result concerns the location of the transition point: a noise-induced bifurcation for the Duffing oscillator occurs precisely where the Lyapunov exponent of the linear oscillator with parametric noise changes sign. We emphasize that this result is genuinely non-perturbative. It results from a simple hypothesis on the behaviour of the conditional probability of the angular variable at fixed energy, valid for very small values of the energy and for *arbitrary* noise amplitude. This assumption is physically plausible (i.e. mean-field-like) and justified by the consequences deduced from it. Having identified the Lyapunov exponent of the stochastic *linear* oscillator as the important physical quantity, we give a simple analytical expression of this exponent as a function of the system's parameters, the damping rate and the noise strength. The location of the transition line that separates the absorbing and oscillatory states of the (nonlinear) Duffing oscillator follows immediately.

Although we have considered here only a cubic oscillator with parametric Gaussian white noise, we believe that the relation between the sign of the Lyapunov exponent and the location of the noise-induced transition holds irrespective of the nonlinearity and of the nature of the noise. Finally, we demonstrate that all physical observables scale linearly with respect to the distance to threshold. Indeed, the argument we use to evaluate the mean energy $\langle E \rangle$ may be easily generalized to show that all moments $\langle E^k \rangle$ also scale linearly, suggesting that the concept of critical exponents is not relevant here. One would certainly like to test experimentally whether noise-induced transitions may indeed be characterized by a multifractal order parameter, at variance with the theory of equilibrium continuous transitions.

Appendix A: Some useful identities for the Lyapunov exponent

In this appendix, we prove that the Lyapunov exponent, defined as $\lim_{t \to \infty} \frac{1}{2t} \overline{\log E}$, satisfies the relation (44). Using the 'radial' variable ρ defined by

$$\rho = \frac{1}{2} \log(2E), \qquad (61)$$

we rewrite equation (20) as a system of nonlinear coupled stochastic equations

$$\dot{\rho} = -\alpha \cos^2 \phi + \sin \phi \cos \phi\, \Xi(t)\,, \qquad (62)$$

$$\dot{\phi} = 1 + \alpha \sin \phi \cos \phi - \sin^2 \phi\, \Xi(t)\,. \qquad (63)$$

The Fokker-Planck evolution equation for the joint probability distribution $P_t(\rho, \phi)$ is then given by [25]

$$\partial_t P_t = \alpha \cos^2 \phi\, \partial_\rho P_t - \partial_\phi \left[(1 + \alpha \sin \phi \cos \phi)\, P_t\right]$$
$$+ \frac{\Delta}{2} \Big\{ \sin^2 \phi \cos^2 \phi\, \partial^2_{\rho\rho} P_t - \sin \phi \cos \phi\, \partial^2_{\rho\phi} \left[\sin^2 \phi\, P_t\right]$$
$$- \partial_\phi \left[\sin^3 \phi \cos \phi\, \partial_\rho P_t\right] + \partial_\phi \left[\sin^2 \phi\, \partial_\phi(\sin^2 \phi\, P_t)\right] \Big\}. \qquad (64)$$

From equation (64) we derive the time evolution of the mean value $\overline{\rho}$, where the overline indicates averaging with respect to $P_t(\rho, \phi)$. After suitable integration by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = -\alpha\, \overline{\cos^2 \phi} + \frac{\Delta}{2}\overline{\sin^2 \phi\, (\sin^2 \phi - \cos^2 \phi)}. \qquad (65)$$

Similarly, we deduce the time evolution of $\overline{\log |\sin \phi|}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\log |\sin \phi|} = \overline{\cotan \phi} + \alpha\, \overline{\cos^2 \phi}$$
$$- \frac{\Delta}{2}\, \overline{\sin^2 \phi\, (\sin^2 \phi - \cos^2 \phi)}. \qquad (66)$$

From equations (65, 66), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = \overline{\cotan \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\overline{\log |\sin \phi|}. \qquad (67)$$

At large times, the PDF $P_t(E, \phi)$ reaches a stationary limit $P_{\mathrm{stat}}(E, \phi)$. Thus, when $t \to \infty$, the averages $\overline{\log |\sin \phi|}$ and $\overline{\cotan \phi}$ have finite limits, given by $\langle \log |\sin \phi| \rangle$ and $\langle \cotan \phi \rangle$, respectively, where the expectation values denoted by brackets $\langle \rangle$ are calculated by using the marginal PDF, $\tilde{P}_{\mathrm{stat}}(\phi)$. Therefore

$$\lim_{t \to \infty} \frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = \langle \cotan \phi \rangle\,. \qquad (68)$$

From equations (61) and (68), we derive

$$\lim_{t \to \infty} \frac{1}{2t}\overline{\log E} = \lim_{t \to \infty} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{\log E} = \lim_{t \to \infty} \frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = \langle \cotan \phi \rangle\,. \qquad (69)$$

Similarly, we take the stationary limit of equation (66) and obtain

$$\langle \cotan \phi \rangle = -\alpha\, \langle \cos^2 \phi \rangle + \frac{\Delta}{2}\, \langle \sin^2 \phi\, (\sin^2 \phi - \cos^2 \phi) \rangle\,. \qquad (70)$$

The right hand side of this equation is nothing but Λ (see Eq. (32)). This concludes the derivation of equation (44):

$$\lim_{t \to \infty} \frac{1}{2t}\overline{\log E} = \langle \cotan \phi \rangle = \Lambda(\alpha, \Delta)\,. \qquad (71)$$

Appendix B: Calculation of the Lyapunov exponent

Expressions for the Lyapunov exponent have been available for a long time in the literature [27], but they are rather unwieldy. In a recent article about the quantum localisation problem [28], an expression for the localisation length is derived: it is shown to be identical to the inverse Lyapunov exponent of the linear oscillator with parametric noise and zero damping. In this appendix, we generalize the calculation of [28] to the dissipative case and derive a particularly simple formula for the Lyapunov exponent.

Since $\Lambda = \langle \cotan\phi \rangle$ (see Eq. (71)), we introduce the auxiliary variable z defined as

$$z = \dot{x}/x = \cotan \phi\,. \qquad (72)$$

Then, equation (20) becomes

$$\dot{z} + 1 + \alpha\, z + z^2 = \Xi(t)\,. \qquad (73)$$

The stationary Fokker-Planck equation associated with equation (73) is given by

$$\frac{\Delta}{2}\, \frac{\mathrm{d}}{\mathrm{d}z} P_{\mathrm{stat}} = -(1 + \alpha z + z^2)P_{\mathrm{stat}} + J\,, \qquad (74)$$

where we have introduced the stationary current J. The solution of this equation is found by the method of the variation of constants:

$$P_{\mathrm{stat}}(z) = \frac{1}{N} \int_{-\infty}^{z} \exp\left\{\frac{2}{\Delta}(\Phi(y) - \Phi(z))\right\} \mathrm{d}y, \qquad (75)$$

where $\Phi(y) = y + \frac{\alpha}{2}y^2 + \frac{1}{3}y^3$. Writing $\int_{-\infty}^{\infty} P_{\mathrm{stat}}(z)\, \mathrm{d}z = 1$, we obtain the normalisation constant N:

$$N = \int_{-\infty}^{+\infty} \mathrm{d}z \int_{-\infty}^{z} \mathrm{d}y \exp\left\{\frac{2}{\Delta}(\Phi(y) - \Phi(z))\right\}. \qquad (76)$$

Following [28], we change the y variable to $u = z - y$ in equation (76):

$$N = \int_{-\infty}^{+\infty} \mathrm{d}z \int_{0}^{+\infty} \mathrm{d}u\, e^{-\frac{2}{\Delta}\left(u - \frac{\alpha}{2}u^2 + \frac{u^3}{3} + z^2 u + z(\alpha u - u^2)\right)}\,. \qquad (77)$$

The integral over z is now Gaussian, and can be evaluated before the integral over u. Hence, we obtain

$$N = \sqrt{\frac{\pi \Delta}{2}}\, \int_{0}^{+\infty} \frac{\mathrm{d}u}{\sqrt{u}}\, \exp\left\{-\frac{2}{\Delta}\left((1 - \frac{\alpha^2}{4})u + \frac{u^3}{12}\right)\right\}\,. \qquad (78)$$

The use of the identity

$$\tilde{P}_{\mathrm{stat}}(\phi)\mathrm{d}\phi = P_{\mathrm{stat}}(z)\mathrm{d}z\,, \qquad (79)$$

yields the stationary angular measure

$$\tilde{P}_{\mathrm{stat}}(\phi) = \frac{1}{N \sin^2 \phi} \int_{-\infty}^{\cotan \phi} \mathrm{d}y\, e^{\frac{2}{\Delta}(\Phi(y) - \Phi(\cotan \phi))}\,. \qquad (80)$$

We deduce that the Lyapunov exponent satisfies

$$\Lambda = \int_0^\pi \tilde{P}_{\mathrm{stat}}(\phi)\cot\phi\,\mathrm{d}\phi \tag{81}$$
$$= \frac{1}{N}\int_{-\infty}^{+\infty}\mathrm{d}z\,z\int_{-\infty}^{z}\exp\left\{\frac{2}{\Delta}(\Phi(y)-\Phi(z))\right\}\mathrm{d}y\,.$$

We again change the variables from (y,z) to (u,z) with $u = z - y$, and obtain

$$\Lambda = \frac{1}{N}\int_{-\infty}^{+\infty}\mathrm{d}z\,z\int_{0}^{+\infty}\mathrm{d}u\,e^{-\frac{2}{\Delta}\left(u-\frac{\alpha}{2}u^2+\frac{u^3}{3}+z(\alpha u-u^2)+z^2 u\right)}\,. \tag{82}$$

Evaluating the Gaussian integral in z leads to the expression of the Lyapunov exponent given in the text

$$\Lambda(\alpha,\Delta) = \frac{1}{2}\left\{\frac{\int_0^{+\infty}\mathrm{d}u\,\sqrt{u}\,e^{-\frac{2}{\Delta}\left((1-\frac{\alpha^2}{4})u+\frac{u^3}{12}\right)}}{\int_0^{+\infty}\frac{\mathrm{d}u}{\sqrt{u}}e^{-\frac{2}{\Delta}\left((1-\frac{\alpha^2}{4})u+\frac{u^3}{12}\right)}} - \alpha\right\}\,. \tag{83}$$

This formula also appears in recent mathematical literature [29]. It constitutes an exact result, obtained without approximations. We have computed numerically the Lyapunov exponent by measuring $\overline{\log E}$ in the time-asymptotic regime and thus confirmed the validity of (83).

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